MATH2050C Assignment 8

Deadline: March 18 , 2025.

Hand in: 4.3. no. 5ab, 11; Supp. Ex. 1, 5, 8.

Section 4.2 no. 11cd, 12; **Section 4.3** no. 3, 4, 5abedh, 8, 11.

Supplementary Problems

1. Find the limits of x^3e^{-x} where $c = -\infty, 0$ and ∞ as $x \to c$.
2. Show that $\lim_{x\to c} \sin x = \sin c$.
3. Show that $x - \frac{x^3}{6} \leq \sin x \leq x$, for $x \in [0,1]$ and deduce $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
4. Find $\lim_{x\to 0} \sin 6x / \sin 5x$.
5. Find the limit of $\sqrt{(x+a)(x+b)} - x$ as $x \to \infty$. Here $a, b > 0$.
6. Evaluate
$$\lim_{x\to -3} \frac{x^2 - 2x - 15}{x+3}.$$
7. Evaluate
$$\lim_{x\to \infty} \frac{\cos 1/x}{x}.$$
8. Find $\lim_{x\to c} \dfrac{5x - \sqrt{x}}{\sqrt{x} - x^3}$ for $c = 0^+$ and ∞.

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Further Comments on Limits of Functions

First, we have studied limits of functions. For polynomials and rational functions, their limits are well understood. Indeed, let $r(x) = p(x)/q(x)$ be a rational function. We knew (1) it is well defined on the set $E = \{x \in \mathbb{R} : \ q(x) \neq 0\}$, (since a polynomial has at most finitely many roots, E is the union of finitely many open intervals.) (2) $\lim_{x\to c} r(x) = r(c)$ whenever c satisfies $q(c) \neq 0$. (Before the evaluation it is better to make sure that p are q are reduced, that is, they do not have common factor.)

In order to have more examples to work on, we need to introduce more functions. In this chapter the following functions are studied:

- The **square root** $f_1(x) = \sqrt{x}$. It is defined on $[0, \infty)$ and $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ for all $c \geq 0$.
- The **(rational) power** $f_2(x) = x^{m/n}$. Generalizing the square root, it is known from the last chapter that for each $x \geq 0$, there is a unique $y \geq 0$ satisfying $y^n = x$. We write $y = x^{1/n}$ the n-th root of x. Then $x^{m/n} = (x^{1/n})^m, x \in [0, \infty)$, is well-defined for all $n, m \in \mathbb{N}$. We also define $x^{-m/n} = 1/x^{m/n}$. The square root function is a special rational power. We have the following general result: Let f be a non-negative function on A and c is a cluster point of A. Then
$$\lim_{x\to c} f(x)^{m/n} = f(c)^{m/n}.$$
- The **absolute value function** $f_3(x) = |f(x)|$. It is defined on $(-\infty, \infty)$ and
$$\lim_{x\to c} |f(x)| = |f(c)| \ ,$$
for all $c \in \mathbb{R}$.
- The **exponential function** $f_4(x) = E(x), x \in \mathbb{R}$. The exponential function is defined to be the limit $E(x) = \lim_{n\to\infty}(1 + x/n)^n$ which was proved to exist previously. It is equal to
$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ .$$
We have
$$\lim_{x\to c} E(x) = E(c) \ ,$$
for all c. We provide a proof of this limit. In the following we adapt the common practise to express $E(x)$ as e^x, although we have only shown this is valid for rational x. First,
$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right| = \frac{x^{n+1}}{(n+1)!} \left| 1 + \frac{x}{n+2} + \frac{x^2}{(n+3)(n+2)} + \cdots \right| \ .$$
Therefore, for $x, |x| \leq M = |c| + 1$, we fix m such that $M/(m+2) \leq 1/2$,
$$\left| e^x - \sum_{k=0}^{m} \frac{x^k}{k!} \right| \leq \frac{M^{m+1}}{(m+1)!} \left(1 + \frac{M}{m+2} + \frac{M^2}{(m+2)^2} + \frac{M^3}{(m+2)^3} + \cdots \right) \leq \frac{2M^{m+1}}{(m+1)!} \ .$$
By the ratio test we see that $M^{m+1}/(m+1)! \to 0$ as $m \to \infty$. For $\varepsilon > 0$, we can further assume m so large that $2M^{m+1}/(m+1)! < \varepsilon/3$. On the other hand, let $p(x) = \sum_{k=0}^{m} x^k/k!$.

As $\lim_{x\to c} p(x) = p(c)$, for $\varepsilon > 0$, there is some δ such that $|p(x)-p(c)| < \varepsilon/3$ for $|x-c| < \delta$. Putting things together, we have

$$\begin{aligned} |e^x - e^c| &= |e^x - p(x) + p(x) - p(c) + p(c) - e^c| \\ &\le |e^x - p(x)| + |p(x) - p(c)| + |p(c) - e^c| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ . \end{aligned}$$

- The **sine function** $f_5(x) = \sin x, x \in \mathbb{R}$. Here the sine function is given by the formula

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \ .$$

  Just like in the case of $E(x)$, one can show that this infinite series is convergent for every x. Similar to the exponential function, we have $\lim_{x\to c} \sin x = \sin c$ for all c. Moreover, we have the inequality $x - x^3/6 \le \sin x \le x$ for $0 \le x \le 1$ which implies

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \ .$$

- The **cosine function** $f_6(x) = \cos x, x \in \mathbb{R}$. Here the cosine function is given by the formula

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \ .$$

  One can show $\lim_{x\to c} \cos x = \cos c$ and

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \ ,$$

  or more precisely,

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

Second, variations on the notion of limits of functions including one-sided limits, divergence at infinity and limits at infinity. Let f be function defined on $(a, b]$. It is said to **tend to** ∞ **(resp.** $-\infty$) at a (**from its right**) if for each $M > 0$, there is some $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) for all $x \in (a, a+\delta)$. The notation is $\lim_{x\to a^+} f(x) = \infty$ (resp. $\lim_{x\to a^+} f(x) = -\infty$). Similarly, one can define $\lim_{x\to b^-} f(x) = \pm\infty$ (limit from the left). For f defined on (a, ∞) (resp. $(-\infty, b)$) we can define $\lim_{x\to\infty} f(x) = L$ if for each $\varepsilon > 0$ there is some $K > 0$ such that $|f(x) - L| < \varepsilon$ for all $x > K$. Similarly, we can define $\lim_{x\to-\infty} f(x) = L$, $\lim_{x\to\infty} f(x) = \pm\infty$, $\lim_{x\to-\infty} f(x) = \pm\infty$, etc. For these variations of limits of functions, the corresponding Sequential Criterion, Limit Theorems, and Squeeze Theorem are for you to explore, or simply look up the text book.